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Eigenvalues and degree deviation in graphs

Vladimir Nikiforov *

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA

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Abstract

Let G be a graph with n vertices and m edges and let $\mu(G) = \mu_1(G) \geqslant \cdots \geqslant \mu_n(G)$ be the eigenvalues of its adjacency matrix. Set $s(G) = \sum_{u \in V(G)} |d(u) - 2m/n|$. We prove that

$$\frac{s^2(G)}{2n^2\sqrt{2m}} \leqslant \mu(G) - \frac{2m}{n} \leqslant \sqrt{s(G)}.$$

In addition we derive similar inequalities for bipartite G.

We also prove that the inequality

$$\mu_k(G) + \mu_{n-k+2}(\overline{G}) \geqslant -1 - 2\sqrt{2s(G)}$$

holds for every $k = 2, \ldots, n$.

Finally we prove that for every graph G of order n,

$$\mu_n(G) + \mu_n(\overline{G}) \leqslant -1 - \frac{s^2(G)}{2n^3}.$$

We show that these inequalities are tight up to a constant factor.

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* Tel./fax: +1 901 678 2416.

E-mail address: vnikifrv@memphis.edu

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1. Introduction

Our notation is standard (e.g., see [3,5,9]); in particular, all graphs are defined on the vertex set $\{1, 2, \ldots, n\} = [n]$ and $G(n, m)$ stands for a graph with n vertices and m edges. We write $\Gamma(u)$ for the set of neighbors of the vertex u and set $d(u) = |\Gamma(u)|$. Given a graph G of order n, we assume that the eigenvalues of the adjacency matrix of G are ordered as $\mu(G) = \mu_1(G) \geqslant \cdots \geqslant \mu_n(G)$. As usual, $\overline{G}$ denotes the complement of a graph G.

Collatz and Sinogowitz [6] showed that $\mu(G) \geqslant 2m/n$ for every graph $G = G(n, m)$. Since equality holds if and only if G is regular, they proposed the value $\epsilon(G) = \mu(G) - 2m/n$ as a relevant measure of irregularity of G. Two other closely related measures of graph irregularity are the functions

$$\operatorname{var}(G) = \frac{1}{n} \sum_{u \in V(G)} \left(d(u) - \frac{2m}{n} \right)^2,$$

$$s(G) = \sum_{u \in V(G)} \left| d(u) - \frac{2m}{n} \right|.$$

Bell [1] compared $\epsilon(G)$ to $\operatorname{var}(G)$ and showed that none of them could be preferred to the other one as a measure of irregularity. He did not, however, give explicit inequalities between $\epsilon(G)$ and $\operatorname{var}(G)$. In this note we prove that for every graph G with n vertices and m edges,

$$\frac{\operatorname{var}(G)}{2\sqrt{2m}} \leqslant \mu(G) - \frac{2m}{n} \leqslant \sqrt{s(G)}. \tag{1}$$

Thus, in view of

$$\frac{s^2(G)}{n^2} \leqslant \operatorname{var}(G) \leqslant s(G),$$

we also have

$$\frac{s^2(G)}{2n^2\sqrt{2m}} \leqslant \mu(G) - \frac{2m}{n} \leqslant \sqrt[4]{n^2 \operatorname{var}(G)}.$$

In addition we derive similar inequalities specifically for bipartite graphs.

Another well-known inequality involving graph eigenvalues is

$$\mu_k(G) + \mu_{n-k+2}(\overline{G}) \leqslant -1 \tag{2}$$

holding for every graph G of order n and every $k = 2, \ldots, n$. Note that if G is regular, equality holds in (2) but the converse is not always true (e.g., for $b > a > 2$, we have $\mu_2(K_{a,b}) + \mu_n\left(\overline{K_{a,b}}\right) = -1$). A natural problem is to find a lower bound on $\mu_k(G) + \mu_{n-k+2}(\overline{G})$ implying explicit equality in (2) for regular G. In this note we show that for every $k = 2, \ldots, n$,

$$\mu_k(G) + \mu_{n-k+2}(\overline{G}) \geqslant -1 - 2\sqrt{2s(G)}. \tag{3}$$

Finally we prove that for every graph G of order n,

$$\mu_n(G) + \mu_n(\overline{G}) \leqslant -1 - \frac{s^2(G)}{2n^3} \tag{4}$$

implying that for any highly irregular graph G, either $\mu_n(G)$ or $\mu_n(\overline{G})$ must be large in absolute value.

We show that inequalities (1), (3), and (4) are tight up to a constant factor. Let us note that these results are readily applicable to the study of quasi-random graph properties.

The rest of the note is organized as follows. In Section 2 we describe algorithms for regularizing graphs with few edge changes. Section 3 contains basic results about spectra of blown-up graphs. In Sections 4–6 we prove inequalities (1), (3), and (4).

2. Efficient regularization

Consider the following natural problem: given a graph G, what is the minimum number of edges $\rho(G)$ that must be changed to obtain a regular graph. Writing $A(G)$ for the adjacency matrix of a graph G, we see that

$$\rho(G) = \frac{1}{2} \min \{\|A(G) - A(R)\|_1 : R \text{ is regular graph of order } v(G)\}.$$

It is almost certain that the problem of estimating $\rho(G)$ has been raised and solved in the literature but, lacking a proper reference, we shall solve it from scratch.

We first show that there exists a graph R^* whose degrees differ by at most one and such that

$$\|A(G) - A(R^*)\|_1 \leqslant s(G).$$

Next we find a regular graph R such that

$$\|A(R) - A(R^*)\|_1 \leqslant 3n.$$

Finally we show that for every graph G,

$$\rho(G) \geqslant \frac{s(G)}{2},$$

implying that our upper bounds on $\rho(G)$ are not too far from the best possible ones.

2.1. Rough regularization

The main result in this section is the following theorem.

Theorem 1. *For every graph $G = G(n, m)$, there exists a graph $R = G(n, m)$ such that $\Delta(R) \leqslant \delta(R) + 1$ and R differs from G in at most $s(G)$ edges. In particular, if $2m/n$ is an integer then R is $(2m/n)$-regular.*

Proof. We shall describe a simple algorithm that produces the graph R by deleting and adding edges to G. Set $d = \lfloor 2m/n \rfloor$.

Step 1

While $\delta(G) < d$ and $\Delta(G) > d + 1$ select u, v with $d(u) = \delta(G)$ and $d(v) = \Delta(G)$. Since $\Gamma(v)\backslash\Gamma(u) \neq \varnothing$, there exists $w \in \Gamma(v)\backslash\Gamma(u)$; delete the edge vw and add the edge uw.

Write G' for the graph obtained upon exiting Step 1. Since Step 1 is iterated as long as $\delta(G) < d$ and $\Delta(G) > d + 1$, we have either $\delta\left(G'\right) = d$ or $\Delta(G') = d + 1$; we may assume $\delta(G') = d$, since the other case is reduced to this one by considering $\overline{G'}$.

If $\Delta(G') \leqslant d + 1$ then terminate the procedure with $R = G'$. Otherwise write A for the set of vertices of degree d, B for the set of vertices of degree $d + 1$, and C for the set of vertices of degree $d + 2$ or higher.

Step 2

While $C \neq \varnothing$ select $u \in A$, $v \in C$. Since $|\Gamma(v)| > |\Gamma(u)|$, we may select $w \in \Gamma(v) \backslash \Gamma(u)$; delete the edge vw and add the edge uw.

Write R for the graph obtained after executing Step 2. Let G', A, B, C be as defined prior to Step 2; set $|A| = k$, $|C| = s$. Each iteration in Step 1 changes two edges and decreases $s(G)$ by 2; therefore, after the execution of Step 1, at most $s(G) - s(G')$ edges of G are changed. Set

$$l = \sum_{u \in C} (d(u) - d - 1).$$

Each iteration in Step 2 changes two edges and decreases l by 1; therefore, there are l iterations in Step 2 and at most $2l$ edges are changed. To complete the proof we have to show that $s(G') \geqslant 2l$. From

$$\begin{aligned} 2m/n &= \frac{1}{n} \sum_{u \in V(G)} d(u) = \frac{1}{n} \sum_{u \in A} d(u) + \frac{1}{n} \sum_{u \in B} d(u) + \frac{1}{n} \sum_{u \in C} d(u) \\ &= \frac{kd + (n-k-s)(d+1) + s(d+1) + l}{n} = d + \frac{n-k+l}{n} \end{aligned}$$

it follows that $k > l$. Furthermore,

$$\begin{aligned} s(G') - 2l &= \sum_{u \in V} \left| d_{G'}(u) - \frac{2m}{n} \right| - 2l \\ &= \sum_{u \in A} \left| d_{G'}(u) - \frac{2m}{n} \right| + \sum_{u \in B} \left| d_{G'}(u) - \frac{2m}{n} \right| + \sum_{u \in C} \left| d_{G'}(u) - \frac{2m}{n} \right| - 2l \\ &= k \frac{n-k+l}{n} + (n-k-s) \frac{k-l}{n} + s \frac{k-l}{n} - l = 2 \frac{(k-l)(n-k)}{n} > 0, \end{aligned}$$

completing the proof. □

2.1.1. *Rough regularization of bipartite graphs*

Call a bipartite graph *semiregular* if vertices belonging to the same color class have equal degrees.

Let G be a bipartite graph A, B be its color classes, and $|A| = a$, $|B| = b$. Define the function

$$s_2(G) = \sum_{u \in A} \left| d(u) - \frac{m}{a} \right| + \sum_{u \in B} \left| d(u) - \frac{m}{b} \right|.$$

For bipartite graphs $s_2(G)$ is more relevant than $s(G)$. Clearly, $s_2(G) = 0$ if and only if G is semiregular.

Modifying slightly the proof of Theorem 1 we obtain the following special case for bipartite graphs.

Theorem 2. *For every bipartite graph $G = G(n, m)$ with color classes A, B, there exists a bipartite graph $R = G(n, m)$ with the same color classes such that*:

(i) $|d_R(u) - d_R(v)| \leqslant 1$ *for every u, v belonging to the same color class*;
(ii) *R differs from G in at most $s_2(G)$ edges.*

In particular, if $m/|A|$ and $m/|B|$ are integers then R is semiregular.

2.2. Fine regularization

If we allow m to change, we may further regularize the graph R obtained in Theorem 1.

Theorem 3. *Let the degrees of a graph $G = G(n, m)$ be either d or $d + 1$. There exists an r-regular graph R such that either $r = d$ or $r = d + 1$, and R differs from G in at most $3n/2$ edges.*

Proof. Write A for the set of vertices of degree $d + 1$ and B for $V(G)\backslash A$. Clearly, either $|A|$ or $|B|$ is even. We shall assume that $|A|$ is even, otherwise we may apply the argument to $\overline{G}$. Set $a = |A|$. Our goal is to construct a d-regular graph by changing at most $3a/2$ edges. We shall describe a procedure constructing R.

Step 1
While $E(A) \neq \varnothing$, select $uv \in E(A)$ and remove it.

Step 2
While $A \neq \varnothing$, select two distinct $u, v \in A$ and two non-adjacent vertices $t \in \Gamma(v)$, $w \in \Gamma(u)$. Delete the edges uw and vt; add the edge wt.

The iteration in Step 2 may always be executed since, for every two distinct $u, v \in A$, there exist non-adjacent vertices $t \in \Gamma(v)$ and $w \in \Gamma(u)$. Indeed, if $\Gamma(u) \neq \Gamma(v)$, select $w \in \Gamma(u)\backslash\Gamma(v)$. Since $d(w) = d < |\Gamma(v)|$, there exists $t \in \Gamma(v)$ that is not joined to w and the assertion is proved. If $\Gamma(u) = \Gamma(v)$ then $\Gamma(u)$ cannot induce a complete graph, since $\Gamma(u) \subset B$ and so the vertices in $\Gamma(u)$ have degree d.

Each iteration in Step 1 removes two vertices from A and changes one edge. Each iteration in Step 2 removes two vertices from A and changes three edges. Therefore, after changing at most $3|A|/2$ edges, we obtain a d-regular graph R, as claimed. □

2.3. Optimal regularization

Summarizing Theorems 1 and 3, we obtain the following corollary.

Corollary 4. *For every graph G of order n,*

$$\rho(G) \leqslant s(G) + \frac{3n}{2}.$$

It turns out that this bound is quite close to the optimal one, no matter what the graph G is. We shall show that

$$\rho(G) \geqslant \frac{s(G)}{2}.$$

Let R be r-regular graph with $V(R) = V(G)$. For every vertex $u \in V(G)$, we have

$$|(\Gamma_G(u)\backslash\Gamma_R(u)) \cup (\Gamma_R(u)\backslash\Gamma_G(u))| \geqslant d(u) + r - 2\min(d(u), r) \geqslant |d(u) - r|.$$

Hence, summing over all vertices $u \in V(G)$, we find that

$$2\rho(G) \geqslant \|A(G) - A(R)\|_1 \geqslant \sum_{u\in V(G)} |d(u) - r| \geqslant s(G)$$

as claimed.

3. The spectra of blown-up graphs

In this section we introduce two operations on graphs and consider how they affect graph spectra.

Let $G = G(n, m)$ and $t > 0$ be integer. Write $G^{(t)}$ for the graph obtained by replacing each vertex $u \in V(G)$ by a set V_u of t vertices and joining $x \in V_u$ to $y \in V_v$ if and only if $uv \in E(G)$. Notice that $v(G^{(t)}) = tn$. The following theorem holds.

Theorem 5. *The eigenvalues of $G^{(t)}$ are $t\mu_1(G), \dots, t\mu_n(G)$ together with $n(t-1)$ additional 0's.*

Set $G^{[t]} = \overline{\overline{G}^{(t)}}$, i.e., $G^{[t]}$ is obtained from $G^{(t)}$ by joining all vertices within V_u for every $u \in V(G)$; note also that $\overline{G^{(t)}} = \overline{G}^{[t]}$. The following theorem holds.

Theorem 6. *The eigenvalues of $G^{[t]}$ are $t\mu_1(G) + t - 1, \dots, t\mu_n(G) + t - 1$ together with $n(t-1)$ additional (-1)'s.*

4. Bounds on $\mu(G)$

In this section we shall prove inequalities (1). Recall first the inequality

$$\mu^2(G) \geqslant \frac{1}{n} \sum_{u \in V(G)} d^2(u) \tag{5}$$

due to Hofmeister [8] and observe that Stanley's inequality [11]

$$\mu(G) \leqslant -\frac{1}{2} + \sqrt{2m + \frac{1}{4}}$$

implies that

$$\mu^2(G) \leqslant 2m. \tag{6}$$

We thus find that

$$2\sqrt{2m}\left(\mu(G) - \frac{2m}{n}\right) \geqslant 2\mu(G)\left(\mu(G) - \frac{2m}{n}\right) \geqslant \mu^2(G) - \left(\frac{2m}{n}\right)^2$$
$$\geqslant \frac{1}{n}\sum_{u \in V(G)} d^2(u) - \left(\frac{2m}{n}\right)^2 = \operatorname{var}(G),$$

obtaining the lower bound in (1). To prove the upper bound we need the following proposition.

Proposition 7. *If G_1 and G_2 are graphs with $V(G_1) = V(G_2)$ then*

$$\mu(G_1) - \mu(G_2) \leqslant \sqrt{2|E(G_1)\backslash E(G_2)|}.$$

Proof. Setting $G' = (V(G_1), E(G_1) \cup E(G_2))$, $G'' = (V(G_1), E(G_1)\backslash E(G_2))$, from Weyl's inequalities [9, p. 181], we have

$$\mu(G_1) \leqslant \mu(G') \leqslant \mu(G_2) + \mu(G'').$$

By (6), we have,

$$\mu\left(G''\right) \leqslant \sqrt{2\left|E\left(G_1\right)\setminus E\left(G_2\right)\right|},$$

completing the proof. □

We shall deduce the upper bound in (1) essentially from Theorem 1.

Theorem 8. *For every graph $G = G(n, m)$,*

$$\mu(G) - \frac{2m}{n} \leqslant \sqrt{s(G)}.$$

Proof. Theorem 1 implies that there exists a graph $R = G(n, m)$ such that $\Delta(R) \leqslant \delta(R) + 1$ and R differs from G in at most $s(G)$ edges. Since $e\,(R) = e(G)$, it follows that $|E\,(G)\setminus E(R)| = |E\,(R)\setminus E(G)|$ and so $2\,|E(G)\setminus E(R)| \leqslant s\,(G)$. Observe that $\Delta(R) \leqslant \delta(R) + 1$ and $e(R) = e(G)$ imply that $\Delta\,(R) = \lceil 2m/n\rceil$, and so, $\mu(R) \leqslant \lceil 2m/n\rceil$. Hence, by Proposition 7,

$$\mu(G) - \frac{2m}{n} \leqslant \mu(G) - \left\lceil \frac{2m}{n} \right\rceil + 1 \leqslant \mu(G) - \mu(R) + 1 \leqslant 1 + \sqrt{s(G)}. \tag{7}$$

Notice that $v\left(G^{(t)}\right) = tn$, $e\left(G^{(t)}\right) = t^2 m$, and $s\left(G^{(t)}\right) = t^2 s(G)$. Applying Theorem 5, we also see that

$$\mu\left(G^{(t)}\right) = t\mu(G).$$

From (7) it follows that

$$(\mu(G) - 2m/n)\, t = \mu(G^{(t)}) - 2e(G^{(t)})/v(G^{(t)}) \leqslant 1 + \sqrt{s(G^{(t)})} = 1 + t\sqrt{s(G)}.$$

Hence, dividing by t and letting t tend to infinity, the desired inequality follows. □

4.1. *Tightness of inequalities (1)*

It is natural to ask how large c could be so that the inequality

$$\mu(G) - \frac{2m}{n} \geqslant c\frac{s^2(G)}{n^2\sqrt{m}} \tag{8}$$

holds for every graph $G = G(n, m)$. Letting $G = K_{n,n+1}$, we see that

$$\mu(G) - \frac{2e(G)}{v(G)} = \sqrt{n\,(n+1)} - \frac{2n\,(n+1)}{2n+1} = \frac{\sqrt{n\,(n+1)}}{(2n+1)\left(2n+1+2\sqrt{n\,(n+1)}\right)},$$

$$\frac{s^2(G)}{v^2(G)\sqrt{e(G)}} = \frac{4n^2\left(n+1-\frac{2n(n+1)}{2n+1}\right)^2}{(2n+1)^2\sqrt{n\,(n+1)}} = \frac{4n^2\,(n+1)^2}{(2n+1)^4\sqrt{n\,(n+1)}},$$

so, if (8) holds for n large enough then it follows that $c \leqslant 1/2$.

Similarly, let c be such that the inequality

$$\mu(G) - \frac{2m}{n} \leqslant c\sqrt{s(G)} \tag{9}$$

holds for every graph $G = G(n, m)$. Letting $G = K_{1,n}$ we see that

$$\mu(G) - \frac{2e(G)}{v(G)} = \sqrt{n} - \frac{2n}{n+1},$$
$$\sqrt{s(G)} = \sqrt{2\frac{n\,(n-1)}{n+1}},$$

so, if (9) holds for n large enough then it follows that $c \geqslant 1/\sqrt{2}$.

We venture the following conjecture.

Conjecture 9. *For every graph G of sufficiently large order n and size m,*

$$\frac{s^2(G)}{2n^2\sqrt{m}} \leqslant \mu(G) - \frac{2m}{n} \leqslant \sqrt{\frac{s(G)}{2}}.$$

4.2. Bounds on $\mu(G)$ when G is bipartite

It is possible to modify inequalities (1) to better suit bipartite graphs.

Let G be a bipartite graph A, B be its color classes, and $|A| = a$, $|B| = b$. Then, by Rayleigh's principle we have

$$\mu(G) \geqslant \frac{e(G)}{\sqrt{ab}}.$$

A careful analysis shows that equality is possible if and only if G is semiregular. In fact the following theorem holds.

Theorem 10. *For every bipartite graph G with color classes A, B,*

$$\frac{s_2^2(G)}{2v^2(G)\sqrt{|A|\,|B|}} \leqslant \mu(G) - \frac{e(G)}{\sqrt{|A|\,|B|}} \leqslant \sqrt{\frac{s_2(G)}{2}}.$$

Proof. Let $|A| = a$, $|B| = b$, $e(G) = m$, $v(G) = n$. We start with the proof of the first inequality. By the AM–QM inequality we have

$$\sum_{u\in A}\left|d(u) - \frac{m}{a}\right| \leqslant \sqrt{a\sum_{u\in A}\left(d(u) - \frac{m}{a}\right)^2},$$
$$\sum_{u\in B}\left|d(u) - \frac{m}{b}\right| \leqslant \sqrt{b\sum_{u\in B}\left(d(u) - \frac{m}{b}\right)^2}.$$

Hence, by Cauchy–Schwarz and inequality (5), we find that,

$$s_2(G) \leqslant \sqrt{n}\sqrt{\sum_{u\in A}\left(d(u) - \frac{m}{a}\right)^2 + \sum_{u\in B}\left(d(u) - \frac{m}{b}\right)^2} = \sqrt{n}\sqrt{\sum_{u\in V(G)} d^2(u) - \frac{m^2 n}{ab}}$$
$$\leqslant n\sqrt{\mu^2(G) - \frac{m^2}{ab}} \leqslant n\sqrt{\left(\mu(G) - \frac{m}{\sqrt{ab}}\right)\left(2\sqrt{ab}\right)}$$

proving the first inequality.

To prove the second inequality we first note the equivalent of Proposition 7 for bipartite graphs: if G_1 and G_2 are bipartite graphs with the same color classes then

$$\mu(G_1) - \mu(G_2) \leqslant \sqrt{|E(G_1) \backslash E(G_2)|}.$$

Note that the coefficient 2 under the square root is missing here, since $\mu(G) \leqslant \sqrt{e(G)}$ for bipartite G (Nosal [10], also [5], p. 86, Corollary).

Theorem 2 implies that there exists a graph $R = G(n, m)$ with color classes A, B such that $|d_R(u) - d_R(v)| \leqslant 1$ for every u, v belonging to the same color class and R differs from G in at most $s_2(G)$ edges. Since $e(R) = e(G)$, it follows that $|E(G)\backslash E(R)| = |E(R)\backslash E(G)|$, and so $2|E(G)\backslash E(R)| \leqslant s_2(G)$. Hence, by Proposition 7,

$$\mu(G) - \mu(R) \leqslant \sqrt{\frac{s_2(G)}{2}}.$$

Applying the inequality $\mu(G) \leqslant \max_{uv \in E(G)} \sqrt{d(u)d(v)}$, due to Berman and Zhang [2], we find that

$$\mu(R) \leqslant \sqrt{\left(\frac{m}{a}+1\right)\left(\frac{m}{b}+1\right)} \leqslant \sqrt{\frac{m^2}{ab}+\frac{mn}{ab}+1} < \frac{m}{\sqrt{ab}} + \sqrt{n+1}$$

and so,

$$\mu(G) - \frac{m}{\sqrt{ab}} \leqslant \sqrt{\frac{s_2(G)}{2}} + \sqrt{n+1}.$$

Now, applying the final argument from the proof of Theorem 8, the desired inequality follows. $\square$

5. A lower bound on $\mu_k(G) + \mu_{n-k+2}\left(\overline{G}\right)$

The main goal of this section is the proof of inequality (3). By Weyl's inequalities ([9], p. 181), for every graph G of order n and every $k = 2, \ldots, n$, we have

$$\mu_k(G) + \mu_{n-k+2}\left(\overline{G}\right) \leqslant \mu_2(K_n) = -1.$$

A simple argument shows that if G is a regular graph then Courant–Fisher's inequalities ([9], p. 179) imply that

$$\mu_k(G) + \mu_{n-k+2}\left(\overline{G}\right) = -1$$

for every $k = 2, \ldots, n$. In fact the following theorem holds.

Theorem 11. *For every $k = 2, \ldots, n$*

$$\mu_k(G) + \mu_{n-k+2}\left(\overline{G}\right) \geqslant -1 - 2\sqrt{2s(G)}.$$

Proof. By Corollary 4 there exists a regular graph R that differs from G in at most $s(G) + 3n/2$ edges. Then, by Weyl's inequalities, for every $k = 2, \ldots, n$,

$$\mu_k(A(G)) + \mu_1(A(R) - A(G)) \geqslant \mu_k(A(R)),$$
$$\mu_{n-k+2}\left(A\left(\overline{G}\right)\right) + \mu_1\left(A\left(\overline{R}\right) - A\left(\overline{G}\right)\right) \geqslant \mu_{n-k+2}\left(A\left(\overline{R}\right)\right).$$

From (6) we see that

$$\mu_1\left(A(R)-A(G)\right) \leqslant \sqrt{\|A(R)-A(G)\|_1} \leqslant \sqrt{2s(G)+3n},$$
$$\mu_1\left(A\left(\overline{R}\right)-A\left(\overline{G}\right)\right) \leqslant \sqrt{\left\|A\left(\overline{R}\right)-A\left(\overline{G}\right)\right\|_1} \leqslant \sqrt{2s(G)+3n},$$

and hence,

$$\mu_k(G)+\mu_{n-k+2}\left(\overline{G}\right) \geqslant \mu_k(R)+\mu_{n-k+2}\left(\overline{R}\right)-2\sqrt{2s(G)+3n}$$
$$= -1-2\sqrt{2s(G)+3n}.$$

Suppose now that t is sufficiently large and consider the graphs $G^{(t)}$ and $\overline{G^{(t)}}$. By Theorem 5 we have

$$\mu_k\left(G^{(t)}\right)=t\mu_k(G).$$

Similarly in view of $\overline{G^{(t)}}=\overline{G}^{[t]}$ and Theorem 6,

$$\mu_{nt-k+2}\left(\overline{G^{(t)}}\right) \leqslant \min\left\{t\mu_{n-k+2}\left(\overline{G}\right)+t-1,-1\right\}.$$

Since, $s\left(G^{(t)}\right)=t^2s(G)$, we see that

$$t\mu_k(G)+t\mu_{n-k+2}\left(\overline{G}\right) \geqslant \mu_k\left(G^{(t)}\right)+\mu_{nt-k+2}\left(\overline{G^{(t)}}\right)-t+1$$
$$\geqslant -t-2\sqrt{2s\left(G^{(t)}\right)+3nt}$$
$$= -t-2t\sqrt{2s(G)+3n/t}.$$

Dividing by t and letting t tend to infinity, we obtain the desired inequality. □

For the graph $G=K_{1,n}$ we have $s(G)=2\frac{n(n-1)}{n+1}$ and $\mu_{n+1}(G)+\mu_2\left(\overline{G}\right)=-1-\sqrt{n}$. Hence,

$$\mu_{n+1}(G)+\mu_2\left(\overline{G}\right)=-1-\left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{s(G)}$$

implying that inequality (3) is tight up to a constant factor less than or equal to 4.

6. An upper bound on $\mu_n(G)+\mu_n\left(\overline{G}\right)$

The main result in this section is the proof of inequality (4); we prove a stronger inequality in Theorem 13 below. We start with an auxiliary result. Given a graph G with vertex set V and $S\subset V$, write $e(S)$ for the number of edges induced by the set S; if $S', S''\subset V$ are disjoint sets, write $e\left(S',S''\right)$ for the number of edges joining a vertex in S' to a vertex in S''.

Lemma 12. *For every graph G of order n, there exists an $\lfloor n/2\rfloor$-set $S\subset V(G)$ such that*

$$e\left(V(G)\backslash S\right)-e(S) \geqslant \frac{1}{4}s(G).$$

Proof. Let $d(1)\leqslant d(2)\leqslant\cdots\leqslant d(n)$ be the degree sequence of G and set $V=[n]$. For every $1\leqslant k\leqslant n$, letting $S=[k]$, we have

$$\sum_{u\in V\backslash S} d(u) - \sum_{u\in S} d(u) = 2e\left(V\backslash S\right) + e\left(S, V\backslash S\right) - 2e\left(S\right) - e\left(S, V\backslash S\right)$$
$$= 2e\left(V\backslash S\right) - 2e\left(S\right).$$

Let a be the median of the numbers $d(1), \ldots, d(n)$. We shall prove that for $S = \lfloor n/2 \rfloor$

$$2e\left(V\backslash S\right) - 2e\left(S\right) \geqslant \sum_{u\in V} |d(u) - a|. \tag{10}$$

Indeed, if n is even, say $n = 2k$, letting $a = (d(k) + d(k+1))/2$, we have

$$\sum_{u\in V\backslash S} d(u) - \sum_{u\in S} d(u) = \sum_{u\in V\backslash S} (d(u) - a) + \sum_{u\in S} (a - d(u)) = \sum_{u\in V} |d(u) - a|,$$

proving (10) for even n.

If n is odd, say $n = 2k+1$, letting $a = d(k+1)$, we have

$$\sum_{u\in V\backslash S} d(u) - \sum_{u\in S} d(u) = \sum_{u\in V\backslash S} (d(u) - a) + \sum_{u\in S} (a - d(u)) + a$$
$$= \sum_{u\in V} |d(u) - a| + a \geqslant \sum_{u\in V} |d(u) - a|,$$

proving (10) for odd n as well.

The assertion of the lemma follows now from (10) and the fact that if a is any real number and b is the mean of the numbers $x_1, \ldots, x_n$ then

$$\sum_{i=1}^{n} |x_i - b| \leqslant \sum_{i=1}^{n} |x_i - a| + n\,|b - a|$$
$$= \sum_{i=1}^{n} |x_i - a| + \left|\sum_{i=1}^{n} (x_i - a)\right| \leqslant 2\sum_{i=1}^{n} |x_i - a|. \qquad \square$$

Theorem 13. *For every graph G of order n,*

$$\mu_n(G) + \mu_n\left(\overline{G}\right) \leqslant -1 - \frac{\left(\sqrt{5} - 1\right) s^2(G)}{2n^3}. \tag{11}$$

Proof. From the interlacing theorem of Haemers (see, e.g., [7,4]), for every bipartition of $V(G) = V_1 \cup V_2$, we have

$$\mu_n(G) \leqslant \frac{e\left(V_1\right)}{|V_1|} + \frac{e\left(V_2\right)}{|V_2|} - \sqrt{\left(\frac{e\left(V_1\right)}{|V_1|} - \frac{e\left(V_2\right)}{|V_2|}\right)^2 + \frac{e\left(V_1, V_2\right)^2}{|V_1|\,|V_2|}}. \tag{12}$$

Assume n even and let $V(G) = V_1 \cup V_2$ be a bipartition such that $|V_1| = |V_2| = n/2$, and $e\left(V_1\right) - e\left(V_2\right) \geqslant s(G)/4$. Letting $e_1 = e\left(V_1\right)$, $e_2 = e\left(V_2\right)$, $e_3 = e\left(V_1, V_2\right)$, $s = s(G)$, from (12), after some simple algebra, we obtain

$$\frac{n}{2}\mu_n(G) \leqslant e_1 + e_2 - \sqrt{\left(e_1 - e_2\right)^2 + e_3^2} \leqslant e_1 + e_2 - \sqrt{\frac{s^2}{16} + e_3^2}. \tag{13}$$

Since $s(G) = s\left(\overline{G}\right)$, we see also that

$$\frac{n}{2}\mu_n\left(\overline{G}\right) \leqslant \binom{n/2}{2} - e_1 + \binom{n/2}{2} - e_2 - \sqrt{\frac{s^2}{16} + \left(\frac{n^2}{4} - e_3\right)^2},$$

and so,

$$\frac{n}{2}\left(\mu_n\left(\overline{G}\right) + \mu_n(G)\right) \leqslant 2\binom{n/2}{2} - \sqrt{\frac{s^2}{16} + e_3^2} - \sqrt{\frac{s^2}{16} + \left(\frac{n^2}{4} - e_3\right)^2}. \tag{14}$$

Observe that, for every positive a, the function $\sqrt{a + x^2}$ is convex. Thus,

$$\sqrt{\frac{s^2}{16} + e_3^2} + \sqrt{\frac{s^2}{16} + \left(\frac{n^2}{4} - e_3\right)^2} \geqslant 2\sqrt{\frac{s^2}{16} + \left(\frac{n^2}{8}\right)^2} = \sqrt{\frac{s^2}{4} + \frac{n^4}{16}},$$

and from (14) we see that

$$\mu_n\left(\overline{G}\right) + \mu_n(G) \leqslant \frac{n}{2} - 1 - \sqrt{\frac{s^2}{n^2} + \frac{n^2}{4}}. \tag{15}$$

Note that by $s(G) < n^2$ we have

$$\left(\frac{\left(\sqrt{5}-1\right)s^2}{2n^3} + \frac{n}{2}\right)^2 \leqslant \left(\frac{\left(\sqrt{5}-1\right)^2}{4} + \frac{\sqrt{5}-1}{2}\right)\frac{s^2}{n^2} + \frac{n^2}{4} = \frac{s^2}{n^2} + \frac{n^2}{4},$$

and so,

$$\sqrt{\frac{s^2}{n^2} + \frac{n^2}{4}} \geqslant \frac{\left(\sqrt{5}-1\right)s^2}{2n^3} + \frac{n}{2}.$$

This together with (15) completes the proof for even n.

To prove the assertion for odd n observe that if t is even, for the graph $G^{(t)}$ we have

$$t\mu_n(G) + t\mu_n\left(\overline{G}\right) + t - 1 = \mu_{tn}\left(G^{(t)}\right) + \mu_{tn}\left(\overline{G^{(t)}}\right) \leqslant -1 - \frac{\left(\sqrt{5}-1\right)t^4s^2}{2t^3n^3}.$$

Dividing by t and letting t tend to infinity, we see that the assertion follows for odd n as well. □

6.1. *Tightness of inequality (11)*

Since the proof of inequality (11) was carried out in several stages, it is reasonable to ask how far from the optimal one is the constant $\left(\sqrt{5}-1\right)/2 = 0.618\ldots$ Let the constant c be such that the inequality

$$\mu_n(G) + \mu_n\left(\overline{G}\right) \leqslant -1 - c\frac{s^2(G)}{n^3}$$

holds for all graphs G of sufficiently large order. We shall prove that then $c \leqslant 8\left(\sqrt{2}-1\right) = 3.313\ldots$

Define a random graph G by taking two disjoint sets A, B of cardinality n, joining every two vertices in A, and joining a vertex in A to a vertex in B with probability $1/2$. We have, almost surely,

$$e(G) = (1 + \mathrm{o}(1))\, n^2, \quad s(G) = (1 + \mathrm{o}(1))\, n^2.$$

By the interlacing theorem of Haemers, almost surely,

$$\mu_1(G) \geqslant \left(\frac{1+\sqrt{2}}{2} + \mathrm{o}(1)\right) n, \quad \mu_1\left(\overline{G}\right) \geqslant \left(\frac{1+\sqrt{2}}{2} + \mathrm{o}(1)\right) n,$$

$$\mu_{2n}(G) \leqslant \left(\frac{1-\sqrt{2}}{2} + \mathrm{o}(1)\right) n, \quad \mu_{2n}\left(\overline{G}\right) \leqslant \left(\frac{1-\sqrt{2}}{2} + \mathrm{o}(1)\right) n.$$

Write $cw_4(H)$ for the number of closed walks on four vertices in H; recall that for every graph H of order n,

$$cw_4(H) = \operatorname{tr}\left(A^4(H)\right) = \sum_{i=1}^{n} \mu_i^4(H).$$

To estimate $cw_4(G)$ it is enough to count the cycles of length 4. Considering the four possible cases of cycles on four vertices in G, we find that, almost surely,

$$cw_4(G) + cw_4\left(\overline{G}\right) = 2cw_4(G) + \mathrm{o}\left(n^4\right) = \frac{17}{4} n^4 + \mathrm{o}\left(n^4\right).$$

Hence,

$$\frac{17}{4} n^4 + \mathrm{o}\left(n^4\right) = \sum_{i=1}^{2n} \left(\mu_i^4(G) + \mu_i^4\left(\overline{G}\right)\right) \geqslant \mu_1^4(G) + \mu_1^4\left(\overline{G}\right) + \mu_{2n}^4(G) + \mu_{2n}^2\left(\overline{G}\right)$$

$$\geqslant 2\left(\frac{1+\sqrt{2}}{2}\right)^4 n^4 + 2\left(\frac{1-\sqrt{2}}{2}\right)^4 n^4 + \mathrm{o}\left(n^4\right) = \frac{17}{4} n^4 + \mathrm{o}\left(n^4\right)$$

implying that, almost surely,

$$\mu_1(G) = \left(\frac{1+\sqrt{2}}{2} + \mathrm{o}(1)\right) n, \quad \mu_1\left(\overline{G}\right) = \left(\frac{1+\sqrt{2}}{2} + \mathrm{o}(1)\right) n,$$

$$\mu_{2n}(G) = \left(\frac{1-\sqrt{2}}{2} + \mathrm{o}(1)\right) n, \quad \mu_{2n}\left(\overline{G}\right) = \left(\frac{1-\sqrt{2}}{2} + \mathrm{o}(1)\right) n.$$

Hence, almost surely,

$$-\left(\sqrt{2} - 1 + \mathrm{o}(1)\right) n = \mu_{2n}(G) + \mu_{2n}\left(\overline{G}\right) \leqslant -c\,(1 + \mathrm{o}(1))\, \frac{n^4}{8n^3}$$

implying the required inequality.

Note also that, for every $k = 2, \ldots, 2n-1$, we have, almost surely,

$$\mu_k(G) + \mu_k\left(\overline{G}\right) = \mathrm{o}(n),$$

thus, Theorem 13 cannot be extended to eigenvalues other than the smallest one.

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References

[1] F. Bell, Note on the irregularity of graphs, Linear Algebra Appl. 161 (1992) 45–54.
[2] A. Berman, X.-D. Zhang, On the spectral radius of graphs with cutvertices, J. Combin. Theory Ser. B 83 (2001) 233–240.
[3] B. Bollobás, Modern Graph Theory, Graduate Texts in Mathematics, Springer-Verlag, New York, 1998, xiv+394 pp.
[4] B. Bollobás, V. Nikiforov, Graphs and Hermitian matrices: eigenvalue interlacing, Discrete Math. 289 (2004) 119–127.
[5] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, VEB Deutscher Verlag der Wissenschaften, Berlin, 1980, 368 pp.
[6] L. Collatz, U. Sinogowitz, Spektren endlicher Grafen, Abh. Math. Sem. Univ. Hamburg 21 (1957) 63–77.
[7] W.H. Haemers, Interlacing eigenvalues and graphs, Linear Algebra Appl. 226/228 (1995) 593–616.
[8] M. Hofmeister, Spectral radius and degree sequence, Math. Nachr. 139 (1988) 37–44.
[9] R. Horn, C. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985, xiii+561 pp.
[10] E. Nosal, Eigenvalues of Graphs, Master's thesis, University of Calgary, 1970.
[11] R. Stanley, A bound on the spectral radius of graphs with *e* edges, Linear Algebra Appl. 87 (1987) 267–269.